# Finite-size scaling in space-time

Michael F. Zimmer
*Department of Physics, University of Houston, Houston, Texas 77204*
*and Institüt für Festkörperforschung, Forschungszentrum Jülich, D-52425 Jülich, Germany*
(Received 21 July 1997)

Finite-size scaling relations are obtained for the time-dependent Ginzburg-Landau equation in a finite space-time volume. Universal ratios above the upper critical dimension (UCD) are demonstrated to hold true under a mild restriction on its aspect ratio. A perturbation expansion is carried out to order one loop below the UCD, demonstrating the validity of the scaling relations. These results are of practical use in determining the dynamic critical exponent via space-time simulations. [S1063-651X(98)13605-X]

PACS number(s): 64.60.Ak, 64.60.Ht, 05.50.+q, 05.70.Jk

## I. INTRODUCTION

As is well known, it is possible for an infinite volume system possessing short-range interactions to experience a continuous phase transition; at this point the correlation length $\xi$ dominates all other scales and determines the long-wavelength behavior. However, when models for these systems are studied computationally, it is always the case that $\xi$ cannot exceed the necessarily finite system size. Thus the critical behavior is eventually precluded.

In 1971 it was shown by Fisher [1] that the critical behavior could nevertheless be studied through the use of finite-size scaling (FSS) relations. These relations show how external scales enter into expressions for thermodynamic variables, and how they can be used to extract the critical behavior from a *finite* system. Within a field theoretic context [2,3], the characteristic scale of the system $L$ does not change the critical exponents. This is because the exponents are due to short distance singularities, which are not affected by finite scales. FSS has been useful in determining the critical exponents, scaling forms, and universal ratios for static thermodynamic variables for a number of systems. It has also been used to determine the scaling form for the relaxation rate of the time-dependent Ginzburg-Landau (TDGL) equation [4]. This was accomplished by identifying the effective action as the path-integral representation of a quantum-mechanical Hamiltonian in imaginary time, and then noting that the relaxation rate was the inverse of the energy gap of the first excited state.

The content of this paper addresses the scaling of space-time averaged quantities, and the perturbative renormalization of the model therein. To date there have been many attempts to measure the dynamical critical exponent $z$ for the TDGL, and as of yet there is only a scattered consensus [5]. By simulating models in a finite space-time volume, and utilizing the scaling relations given here, a new opportunity is available for determining this exponent. Indeed, the results of this paper have already been applied towards that, in a Monte Carlo simulation of a $2+1$ Ising model [6]. In addition to the scaling relations, a perturbation calculation of the TDGL on a space-time lattice is made, and the renormalizability is explicitly confirmed (to order one loop). This conforms to a proof from Zinn-Justin [7] that the form of the Langevin equation is maintained despite renormalizations. This result relies on Ward-Takahashi identities which follow from the supersymmetry of the effective action. Also, this proof is necessary since standard renormalization group (RG) arguments only provide for the renormalizability of the model, and do not offer enough constraints on the many interactions to ensure that the form of the Langevin equation is maintained. Finally, it should be noted that supersymmetry ensures that in the long-time limit the correct static correlations are recovered.

The path-integral formulation of a nonlinear Langevin equation is reviewed in Sec. II. It is demonstrated how a constrained theory can be made to appear as an equilibrium field theory, up to a Jacobian of transformation. In Sec. III, the outline of the renormalization procedure is reviewed, both for completeness and to demonstrate how a finite spatial and temporal extent affect the scaling. Above four dimensions the fluctuations may be neglected in calculating the FSS relations, as is shown in Sec. IV. Fluctuations are treated in Sec. V; modes which depend on nonzero wave numbers and frequencies are integrated out and their effect on the zero mode is found. Many of the details are relegated to Appendixes A and B. Finally, concluding remarks are given in Sec. VI.

## II. FORMULATION

The time-dependent Ginzburg-Landau equation [8] is frequently invoked to describe the critical behavior of a variety of systems. As an example, it can be used to describe the dynamics of the nonconserved magnetization $\varphi$ through the equation of motion:

$$\partial_t \varphi = -\Omega_0 F[\varphi] + \nu, \tag{1}$$

$$F[\varphi] = (\tau_0 - \nabla^2)\varphi + u_0 \varphi^3, \tag{2}$$

where $\nu$ is a zero-mean random field with autocorrelation

$$\langle \nu(x)\nu(x')\rangle = 2\Omega_0 \delta(x-x'), \tag{3}$$

where $x \equiv (\vec{r},t)$, and $\langle\,\rangle$ represents an average over noise (this ensures the fluctuation dissipation theorem of the second kind is satisfied). The parameter $\tau_0$ is the bare reduced temperature, $u_0$ the bare coupling constant, and $\Omega_0$ is proportional to the bare relaxation rate of a single spin. Periodic boundary conditions in space and time are used, although more complicated generalizations may be used as well. We will not explicitly introduce any short distance scale to regularize the theory, rather a dimensional regularization will be assumed throughout.

The goal is to determine the statistical properties of $O[\varphi_\nu]$, a polynomial in the field $\varphi_\nu$ [i.e., the solution to Eq. (1)]. In the usual way, an expectation value is rewritten as

$$\langle O[\varphi]\rangle = \left\langle \int \mathcal{D}g\,\delta[g[\varphi]]O[\varphi]\right\rangle, \tag{4}$$

where

$$g[\varphi] \equiv \dot{\varphi} + \Omega_0 F - \nu. \tag{5}$$

This may be rewritten as

$$\langle O[\varphi]\rangle = \left\langle \int \mathcal{D}\varphi \;\det \mathbf{M}\delta[\dot{\varphi}+\Omega_0 F-\nu]O[\varphi]\right\rangle, \tag{6}$$

where $\mathbf{M}=\partial g/\partial\varphi$. After introducing an auxiliary field $\tilde{\varphi}$ to exponentiate the $\delta$ function constraint, and then averaging over $\nu$,

$$\langle O[\varphi]\rangle = \int \mathcal{D}\varphi\mathcal{D}\left(\frac{\tilde{\varphi}}{2\pi}\right)\det \mathbf{M}O[\varphi]\exp\{-A\}, \tag{7}$$

where the action $A$ is

$$A = \int dx\{i\tilde{\varphi}(\dot{\varphi}+\Omega_0 F) - \Omega_0\tilde{\varphi}^2\} \tag{8}$$

and $\int dx$ denotes $\int d^d r\ dt$ ($d$ is the spatial dimensionality). In the following, it will be more convenient to simply focus on the generating functional

$$\mathcal{Z}[J,\tilde{J}] = \int \mathcal{D}\varphi\mathcal{D}\left(\frac{\tilde{\varphi}}{2\pi}\right)\det \mathbf{M} \times\exp\left\{-A+\int dx(J\varphi+\tilde{J}\tilde{\varphi})\right\}, \tag{9}$$

so that by differentiation with respect to $J$, $\tilde{J}$, an expression such as $\langle O[\varphi]\rangle$ may be recovered.

Prior to averaging over the noise, the action density appears as $\tilde{\varphi}(\dot{\varphi}+\Omega_0 F-\nu)$, which by the method of stationary phases (or the definition of the functional $\delta$ function) simply enforces the original equation of motion. Upon averaging over $\nu$ and integrating over $\tilde{\varphi}$ the action density becomes $-(\dot{\varphi}+\Omega_0 F)^2/4\Omega_0$, which shows that the deterministic equation $\partial_t\varphi=-\Omega_0 F$ is enforced, up to Gaussian deviations. Physically, this is the nature of the path-integral weight $e^{-A}$, statistically favoring those histories which most nearly satisfy the equation of motion (without noise). However, in the following the field $\tilde{\varphi}$ will be retained, as it helps significantly in the bookkeeping. [The Jacobian cancels a set of graphs that are proportional to $\theta(0)$, the Heaviside function with zero argument].

## III. OUTLINE OF RENORMALIZATION

In this section renormalization group equations are given in the case when there is a characteristic length scale $L$ and time scale $T$. The model is defined on a continuous space-time volume $L^dT$, that is, a spatial hypercube of $d$ dimensions extended over a time $T$. The ultraviolet divergences that are normally dealt with in this context arise, of course, from the behavior of the theory at small lengths and times. Thus the presence of the scales $L$ and $T$ do not affect the usual divergences present in the infinite volume case, and what is more, these scales need not be renormalized themselves [2].

The RG equation may be derived in the usual way for the correlation function

$$\langle \varphi^N\tilde{\varphi}^{\tilde{N}}\rangle = G^{(N,\tilde{N})}\left(\vec{r},\frac{t}{T};g,\tau,\Omega T,M,L,\mu\right), \tag{10}$$

where $\mu$ is a momentum scale and $g=\mu^{-\epsilon}u$ is the dimensionless coupling ($\epsilon=4-d$). The equation is found to be

$$\left\{\mu\frac{\partial}{\partial\mu}+\beta(g)\,\frac{\partial}{\partial g}+\frac{\gamma(g)}{2}\left(N-M\frac{\partial}{\partial M}\right)+\frac{\tilde{N}}{2}\,\tilde{\gamma}(g) - \gamma_\tau(g)\tau\frac{\partial}{\partial\tau}-\gamma_\Omega(g)\frac{\partial}{\partial\Omega}\right\}G^{(N,\tilde{N})}=0. \tag{11}$$

In solving this equation with the method of characteristics, a dimensionless scaling parameter $\rho$ enters in a way parallel to $\mu$. Thus $\mu$ is replaced by $\mu\rho$, and $\rho\sim 0$ corresponds to the critical regime, while $\rho=1$ is the ''initial condition'' [for $\epsilon>0$ and $\beta(g=g^*\neq 0)=0$]. As a function of $\rho$ the parameters obey the following flow equations (i.e., the Wilson equations):

$$\rho\frac{d\tilde{Z}_\varphi(\rho)}{\rho} = \tilde{\gamma}(\rho)\tilde{Z}_\varphi(\rho), \quad \tilde{Z}_\varphi(1)=1, \tag{12}$$

$$\rho\frac{dZ_\varphi(\rho)}{\rho} = \gamma(\rho)Z_\varphi(\rho), \quad Z_\varphi(1)=1, \tag{13}$$

$$\rho\frac{\partial\tau(\rho)}{\partial\rho} = -\gamma_\tau(\rho)\tau(\rho), \quad \tau(1)=\tau, \tag{14}$$

$$\rho\frac{\partial\Omega(\rho)}{\partial\rho} = -\gamma_\Omega(\rho)\Omega(\rho), \quad \Omega(1)=\Omega, \tag{15}$$

$$\rho\frac{\partial g(\rho)}{\partial\rho} = \beta(\rho), \quad g(1)=g, \tag{16}$$

$$M(\rho)=\mathcal{Z}_\varphi^{-1/2}(\rho)M. \tag{17}$$

Solving for the scaling solution of $G^{(N,\tilde{N})}$, and then scaling out $\mu\rho$ gives

$$G^{(N,\tilde{N})}(g,\tau,\Omega T,M,L,\mu)$$
$$\sim \rho^{\gamma^*_{N\tilde{N}}}(\mu\rho)^{d_{N\tilde{N}}} G^{(N,\tilde{N})}\Bigg(g(\rho),\frac{\tau(\rho)}{\mu^2\rho^2},\Omega(\rho)T\mu^2\rho^2,M(\rho)\times(\mu\rho)^{-1+\epsilon/2},L\mu\rho,1\Bigg), \tag{18}$$

where

$$\gamma^*_{N\tilde{N}}=\frac{N}{2}\eta+\frac{\tilde{N}}{2}(\eta+2z-4), \tag{19}$$

$$d_{N\tilde{N}}=N\left(-1+\frac{d}{2}\right)+\tilde{N}\left(1+\frac{d}{2}\right). \tag{20}$$

Taking $L\mu\rho=1$, so that the infrared limit $\rho\to 0$ is approached as $L\mu\to\infty$, it follows that

$$\Omega(\rho)T(\mu\rho)^2\sim T\Omega\rho^{-\gamma^\star_\Omega}L^{-2}=\mu^2\Omega T(L\mu)^{-z}, \tag{21}$$

$$\frac{\tau(\rho)}{\mu^2\rho^2}\sim\frac{\rho^{-\gamma^\star_\tau}\tau}{\mu^2\rho^2}=\frac{\tau}{\mu^2}(L\mu)^{1/\nu}, \tag{22}$$

$$\frac{M(\rho)}{(\mu\rho)^{1-\epsilon/2}}\sim\frac{M}{\mu^{1-\epsilon/2}}(L\mu)^{\beta/\nu}. \tag{23}$$

In summary, what has been shown is that near the critical point

$$G^{(N,\tilde{N})}\left(\vec{r},\frac{t}{T};g,\tau,\Omega T,M,L,\mu\right)\sim L^{-\gamma^*_{N\tilde{N}}-d_{N\tilde{N}}}\times G^{(N,\tilde{N})}\left(\frac{\vec{r}}{L},\frac{t}{T};g^*,\tau L^{1/\nu},\frac{\Omega T}{L^z},ML^{\beta/\nu},1,1\right). \tag{24}$$

Thus the strongest statement we can make just from this renormalization scheme is that the scaling function will take this form, having two separate scaling variables. To go further, explicit scaling functions must be calculated; this will be pursued starting at the mean-field level. Finally, it should be emphasized that these equations only tell how the parameters will be renormalized, not whether the form of the Langevin equation will be maintained. Working from a supersymmetrical form of the action, it is possible to prove the latter [7].

### IV. MEAN FIELD ($d>4$)

To study the scaling behavior above the critical dimension of 4, it suffices to simply neglect all fields that have a nonzero dependence on $p=(\vec{k},\omega)$. Setting $\varphi=\varphi_{p=0}$, a field average may be written as

$$\langle\varphi^{2n}\rangle=\int_{-\infty}^{\infty}d\varphi\int_{-\infty}^{\infty}\frac{d\tilde{\varphi}}{2\pi}V\Omega_0(\tau_0+3u_0\varphi^2)\varphi^{2n}\times\exp\{V\Omega_0\tilde{\varphi}(i\tau\varphi+iu_0\varphi^3-\tilde{\varphi})\}, \tag{25}$$

where the normalization follows from the requirement $\langle 1\rangle=1$. If the fields are scaled as

$$\varphi\to(u_0\sqrt{\Omega_0 V})^{-1/3}\varphi, \tag{26}$$

$$\tilde{\varphi}\to(\Omega_0 V)^{-1/2}\tilde{\varphi}, \tag{27}$$

and $\tilde{\varphi}$ is integrated out, then

$$\langle\varphi^{2n}\rangle=(u_0\sqrt{\Omega_0 V})^{-2n/3}f_{2n}\left[\tau_0\left(\frac{\Omega_0 V}{u_0}\right)^{1/3}\right], \tag{28}$$

$$f_{2n}(x)=\frac{1}{\sqrt{\pi}}\int_0^\infty d\varphi(x+3\varphi^2)\varphi^{2n}\exp\{-(x\varphi+\varphi^3)^2/4\}. \tag{29}$$

The RG equation derived in an earlier section tells us how to find the renormalized form of the above equation: $L$ is set to 1, and the bare parameters are replaced by their renormalized version and made dimensionless with $L$. For example, $\tau_0$ is replaced by $\tau L^{1/\nu}$. Repeating this with all the parameters leads to

$$\left[L^{\beta/\nu}\left(\frac{\Omega T}{L^z}\right)^{1/6}\right]^{2n}\langle\varphi^{2n}\rangle=\bar{f}_{2n}\left[\tau L^{1/\nu}\left(\frac{\Omega T}{L^z}\right)^{1/3}\right]. \tag{30}$$

This is one of the main results of this paper. The above equation can be used to find, for example, how averaged moments of the magnetization scale with $L$ and $T$. In turn, this can be used to determine the dynamic critical exponent $z$, as was done in Ref. [6]. It is important to note that the "shape factor" $\Omega T/L^z$ defines a universality class for this space-time (ST) model, in analogy to characteristic ratios for other anisotropic systems [9]. Also, when this ratio is unity, the effect of the ST volume seems to disappear. This makes some physical sense, since one can imagine that a correlated ST region would be least disturbed by this geometry, as the ratios $\xi^z/T$ and $\xi/L$ would be proportional (near $T_c$).

Following Brézin and Zinn-Justin, many universal, dimensionless ratios may be deduced at the critical point ($\tau_0=0$), for example,

$$\frac{\langle\varphi^4\rangle}{\langle\varphi^2\rangle^2}=\sqrt{\pi}\,\frac{\Gamma(\frac{7}{6})}{\Gamma(\frac{5}{6})^2}=1.290\,54\ldots, \tag{31}$$

which should be compared to $1.10\pm.05$ as found on a two dimensional Ising model by Monte Carlo simulation [6]. There are two classes of corrections to this mean-field result that must be considered; they are due to tree and one-loop terms. In this ST model there is, however, the additional complication due to the Jacobian. Its presence gives rise to interactions which serve to cancel graphs with a closed loop of $\langle\varphi\tilde{\varphi}\rangle$ propagators, as well as to ensure that the parameters in the Jacobian and the action are renormalized in a consistent way (i.e., that the condition $Z[0,0]=1$ is preserved, order by order). The tree terms have the form

$$(\Omega_0 V)\varphi^n\tilde{\varphi}^m, \tag{32}$$

which upon scaling as in Eq. (27) behave as

$$(\Omega_0 V)^{1-n/6-m/2}. \tag{33}$$

This decays with $V$ so long as $n>6-3m$; thus the RG-irrelevant terms may be ignored at tree level. The quartic coupling leads to one-loop graphs, which are proportional to $(\varphi\tilde{\varphi})^{N_1}(\varphi^2)^{N_2}$. Upon scaling as in Eqs. (26) and (27), this behaves as (with $N_1 \geqslant 1$, $N_2 \geqslant 0$)

$$\left(\frac{L^6}{\Omega_0 V}\right)^{(2N_1+N_2)/3}\left(\frac{\Omega_0 T}{L^2}\right). \tag{34}$$

Thus if $\Omega_0 T \sim L^2$, they will decay for $d>4$. Also, if $d=4$, the graphs begin to decay when $\Omega T$ increases faster than $L^2$ for $N_2>1$ (and is marginal for $N_2=1$, $N_1=1$). The choice of $\Omega_0 T \sim L^2$ was taken in Ref. [6] for other reasons; it will also be assumed here.

## V. FLUCTUATION EFFECTS

After Fourier transforming Eq. (5), the constraint for $\varphi_p$ becomes

$$g_p = V\left((-i\omega+\omega_k)\varphi_p + u_0\Omega_0\sum_{p_i}\varphi_{p_1}\varphi_{p_2}\varphi_{p_3}\delta_{\Sigma p_i,p} - \nu_p\right), \tag{35}$$

where

$$\vec{k} = \frac{2\pi}{L}\vec{n}, \quad \vec{n}\in\mathcal{Z}^d \tag{36}$$

$$\omega = \frac{2\pi}{T}m, \quad m\in\mathcal{Z}. \tag{37}$$

As discussed earlier, an auxiliary field may be introduced and the noise averaged over, to give the generating functional (sans sources):

$$Z = \int \mathcal{D}\varphi\mathcal{D}\left(\frac{\tilde{\varphi}}{2\pi}\right)\det M e^{-A}, \tag{38}$$

$$A = -\sum_p V\Big\{ i\tilde{\varphi}_{-p}\Big((-i\omega+\omega_k)\varphi_p + u_0\Omega_0\sum_{p_i}\varphi_{p_1}\varphi_{p_2}\varphi_{p_3}\delta_{\Sigma p_i,p}\Big) - \Omega_0\tilde{\varphi}_{-p}\tilde{\varphi}_p\Big\}, \tag{39}$$

with $\omega_k = \Omega_0(\tau_0+\vec{k}^2)$.

The effect of the modes with nonzero $p$ dependence will be to renormalize the values of the parameters, as well as the fields with zero $p$ dependence. There will also be terms that can be shown to be irrelevant in the RG sense, and so will be ignored. Thus, integrating over all fields that depend on nonzero $p$ (see Appendix A), it is found that $\tau_0$ and $u_0$ take on the effective values

$$\tau_0' = \tau_0 + 6u_0 S_1 + O(u_0^2),$$

$$u_0' = u_0 - 18u_0^2 S_2 + O(u_0^2), \tag{40}$$

where

$$S_1 = \frac{\Omega_0}{V}\sum_p{}' \frac{1}{(\omega^2+\omega_k^2)}, \tag{41}$$

$$S_2 = \frac{\Omega_0^2}{V}\sum_p{}' \frac{2\omega_k}{(\omega^2+\omega_k^2)^2}, \tag{42}$$

and the prime indicates that $p=0$ is not summed over. These sums may be evaluated by dimensional regularization (see Appendix B), and the divergences may be absorbed with a minimal subtraction scheme by defining (with $g_0=\mathcal{Z}_g g$, $\tau_0=\mathcal{Z}_\tau \tau$)

$$Z_\tau = 1 + \frac{6}{\epsilon}g + O(g^2), \tag{43}$$

$$Z_g = 1 + \frac{18}{\epsilon}g + O(g^2), \tag{44}$$

which are the same renormalizations in the static $\varphi^4$ model [8]. Because these are renormalizations of *static* parameters, it is not surprising that they are the same. Within the context of equilibrium Langevin models this was shown by De Dominicis [10] to always be the case. In any event, as the critical temperature is approached, the coupling constant approaches its critical value ($g^*=\epsilon/6$), and gives rise to the same anomalous dimensions found in the infinite volume case.

The effect of the renormalization on the universal ratio of the preceding section may be found by substituting renormalized values for $\tau_0$ and $u_0$ in the argument of Eq. (29). Of course, if this is done at the new critical temperature (i.e., $\tau=0$), then the universal constant will not be changed. If, however, it is possible to determine the bare critical temperature (e.g., from a Langevin model), then with respect to that it is possible to find how the fluctuations affect the universal ratio. Working at $\tau_0=0$, the new (renormalized) argument of $f_{2n}$ in Eq. (29) becomes [with $\hat{g}=(\mu L)^\epsilon g^*$]

$$y = \frac{\hat{g}}{(16\pi^2)^{1/3}}\left(\frac{\Omega T}{L^2}\frac{1}{\hat{g}}\right)^{1/3}\left\{8\pi^2\left(\frac{\Omega T}{L^2}\right)+12I_1\right\}, \tag{45}$$

$$I_1(\tau_0=0) = -5.545\,17\ldots, \tag{46}$$

where $I_1$ is defined in Eq. (B9). It must be remembered that at this point the parameters $\tau$ and $\Omega$ are evaluated at the critical point, which means they now depend on $L$ [i.e., $\tau(L)L^2 \sim \tau L^{1/\nu}$ and $\Omega(L)T/L^2 \sim \Omega T/L^z$]. However, this is irrelevant for $z$ at this order, since $z=2+O(\epsilon^2)$. As a function of $y$, the dimensionless ratio of Eq. (31) becomes

$$\frac{\langle\varphi^4\rangle}{\langle\varphi^2\rangle^2} = \sqrt{\pi}\,\frac{\Gamma(7/6)}{\Gamma(5/6)^2}\left\{1+y\,\frac{2^{4/3}}{3}\left(\frac{\Gamma(1/2)}{\Gamma(5/6)}-\frac{\Gamma(5/6)}{\Gamma(7/6)}\right)\right\} + O(y^2). \tag{47}$$

## VI. CONCLUDING REMARKS

Finite-size scaling relations have been obtained for the time-dependent Ginzburg-Landau equation when it is constrained in time as well as space. This has already proven useful in Monte Carlo simulations [6] that were performed on finite spatial volumes, towards the end of determining the dynamical critical exponent for the model. It should be noted that the method of the calculation used here may be applied to other models as well. In the case of nonequilibrium models, such as those with an additional scale due to a time-dependent field, the current picture could potentially be enriched.

The general approach and several technical details may be traced back to works of Brézin and Zinn-Justin [2], where the effect of a finite spatial scale was calculated. In addition, Zinn-Justin was able to use the supersymmetry of the effective action to prove that the form of the Langevin equation is maintained under renormalization [7]. Also, a scaling form for the relaxation rate has already been derived by Goldschmidt and Niel and Zinn-Justin [4], in which they took advantage of the similarity of the effective action to a quantum-mechanical Hamiltonian.

While the main result was the scaling behavior of the magnetization, the main technical difficulty in this paper was due to the Jacobian, whose presence is a consequence of this being a (dynamically) constrained theory. Thus the results given here could perhaps be carried over to gauge theories, for example. Physical considerations and certain technical aspects imply that the choice of the temporal extent $T \sim L^2$ is special. But this is just a consequence of the mean-field value of the dynamical critical exponent being equal to 2. The importance of the aspect ratio for systems with more than one correlation length has also been noted in purely spatial models [9].

## APPENDIX A

The elements of the determinant are

$$\frac{\delta g_p}{\delta \varphi_{p'}} = V(-i\omega+\omega_k)\delta_{p,p'} + 3u_0\Omega_0 V \sum_{qq'} \varphi_q \varphi_{q'} \delta_{q+q',p-p'} \equiv (\mathbf{G}+\mathbf{B})_{p',p}, \tag{A1}$$

where $\mathbf{G}$ is the free part and $\mathbf{B}$ is proportional to $u_0$. The determinant may be rewritten as

$$\det(\mathbf{G}+\mathbf{B}) = \exp\{\mathrm{tr}\ \ln(\mathbf{G}+\mathbf{B})\} \tag{A2}$$

$$= \left\{\prod_p V(-i\omega+\omega_k)\right\} \exp\left\{\mathrm{tr}(\mathbf{G}^{-1}\mathbf{B}) - \frac{1}{2}\mathrm{tr}(\mathbf{G}^{-1}\mathbf{B})^2 + \cdots\right\}. \tag{A3}$$

In further detail, the trace terms are

$$\mathrm{tr}(\mathbf{G}^{-1}\mathbf{B}) = -3iu_0\Omega_0 V \sum_q \langle \tilde{\varphi}_{-q}\varphi_q\rangle \sum_p \varphi_p \varphi_{-p}, \tag{A4}$$

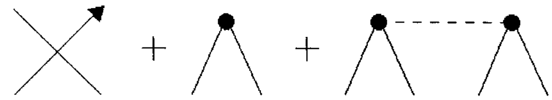

FIG. 1. Diagrammatic representation of the interaction vertices $-A_I'$.

$$\mathrm{tr}(\mathbf{G}^{-1}\mathbf{B})^2 = -(3u_0\Omega_0 V)^2 \sum_{q,q'} \langle \tilde{\varphi}_{-q}\varphi_q\rangle \langle \tilde{\varphi}_{-q'}\varphi_{q'}\rangle \times \left(\sum_{p,p'} \varphi_p\varphi_{p'}\delta_{p+p',q-q'}\right) \times \left(\sum_{p,p'} \varphi_p\varphi_{p'}\delta_{p+p',q'-q}\right). \tag{A5}$$

Fields with nonzero $p$ dependence will be integrated over in the generating functional:

$$Z = \int \mathcal{D}\varphi \mathcal{D}\left(\frac{\tilde{\varphi}}{2\pi}\right) \det M_0 \exp\{-A - A' - A_I'\}, \tag{A6}$$

$$\mathcal{D}\varphi\mathcal{D}\left(\frac{\tilde{\varphi}}{2\pi}\right) = \frac{d\varphi d\tilde{\varphi}}{2\pi} \times \prod_{p>0} \left(\frac{d\ \mathrm{Re}\ \varphi_p d\ \mathrm{Im}\ \varphi_p}{\pi} \frac{d\ \mathrm{Re}\ \tilde{\varphi}_p d\ \mathrm{Im}\ \tilde{\varphi}_p}{\pi}\right), \tag{A7}$$

$$\det M_0 = \prod_p V(-i\omega+\omega_k), \tag{A8}$$

$$-A = V\{i\tilde{\varphi}(\omega_0\varphi + u_0\Omega_0\varphi^3) - \Omega_0\tilde{\varphi}^2\}, \tag{A9}$$

$$-A' = V\{i\tilde{\varphi}_{-p}(-i\omega+\omega_k)\varphi_p - \Omega_0\tilde{\varphi}_{-p}\tilde{\varphi}_p\}, \tag{A10}$$

$$-A_I' = iu_0\Omega_0 {\sum_{p_i}}' \tilde{\varphi}_{p_1}\varphi_{p_2}\varphi_{p_3}\varphi_4 \delta_{\Sigma p_i,0} + \mathrm{tr}(\mathbf{G}^{-1}\mathbf{B}) - \frac{1}{2}\mathrm{tr}(\mathbf{G}^{-1}\mathbf{B})^2 + O(u_0^3). \tag{A11}$$

The three interaction terms on the right hand side are represented diagrammatically in Fig. 1; a straight line represents $\varphi$, and an arrow represents $\tilde{\varphi}$. After integrating over all fields with nonzero $p$ dependence, the result is

$$Z = \int d\varphi d\tilde{\varphi}\omega_0 V(1+\Delta)e^{-A}, \tag{A12}$$

$$\Delta = 3u_0\left(\frac{\Omega_0}{\omega_0}\right)\varphi^2 + u_0 S_1\left\{6\left(\frac{\Omega_0}{\omega_0}\right) + 6i\Omega_0 V\varphi\tilde{\varphi}\right\} \tag{A13}$$

$$-u_0^2 S_2\left\{54\left(\frac{\Omega_0}{\omega_0}\right)\varphi^2 + 18i\Omega_0 V\varphi^3\tilde{\varphi}\right\} + O(u_0^3), \tag{A14}$$

where $S_1$ and $S_2$ are the same as in Eqs. (41 and 42). The diagrammatic expression of $\Delta$ is shown in Fig. 2. It is ex-

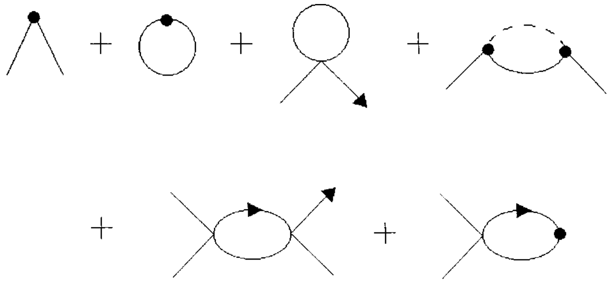

FIG. 2. Diagrammatic representation of the graphs Δ.

pected that Δ is due to a renormalization of $A$ and of $\omega_0 V$. So allowing for shifts of these two quantities

$$\omega_0 V \to \omega_0 V + \Delta_1, \tag{A15}$$

$$-A \to -A + \Delta_2, \tag{A16}$$

and imposing the normalization conditions that the action be proportional to $\tilde{\varphi}$ and that there be no constant terms, it follows that

$$\omega_0 V \to \Omega_0 V\{\tau_0 + 6u_0 S_1 + 3\varphi^2(u_0 - 18u_0^2 S_2)\}, \tag{A17}$$

$$-A \to \Omega_0 V\{i\varphi\tilde{\varphi}(\tau_0 + 6u_0 S_1) + i\tilde{\varphi}\varphi^3(u_0 - 18u_0^2 S_2) - \tilde{\varphi}^2\}. \tag{A18}$$

Redefining $u_0$ and $\tau_0$ to the effective values in Eq. (40), $\omega_0 V$ and $A$ retain their original form, and the model is seen to be renormalizable (to this order).

## APPENDIX B

The sums given in Eqs. (41) and (42) may be evaluated with the identity

$$\coth(\pi x) = \frac{1}{\pi x} + \frac{2x}{\pi} \sum_{m=1}^{\infty} \frac{1}{x^2 + m^2}. \tag{B1}$$

Upon summing over the frequency, it follows that

$$S_1 = \frac{1}{12}\frac{\Omega_0 T}{L^d} + \frac{1}{2}\frac{\Omega_0}{L^d}\sum_{\vec{k}}{}' \frac{1}{\omega_k} + \cdots, \tag{B2}$$

$$S_2 = \frac{1}{2}\frac{\Omega_0^2 T}{V}\sum_{\vec{k}}{}' \frac{1}{\omega_k^2}, \tag{B3}$$

where $\omega_k = \Omega_0(\tau_0 + \vec{k}^2)$ and the dots represent exponentially small corrections (that can be ignored). Aside from the constant shift added to $S_1$, these sums are the same as in the $\varphi^4$ model, as was expected. The divergences may be extracted from the sums by making use of the identity (for $N=1,2$)

$$\sum_{\vec{n}}{}' \frac{1}{(a+\vec{n}^2)^N} = \int_0^\infty e^{-as} s^{N-1}\{A(s)^d - 1\} ds, \tag{B4}$$

where

$$A(s) = \sum_{m \in \mathcal{Z}} e^{-sm^2}. \tag{B5}$$

The Poisson formula allows one to show that

$$A(s) = \sqrt{\frac{\pi}{s}}\, A\left(\frac{\pi^2}{s}\right), \tag{B6}$$

which makes it clear that there are divergences in the integrals as $s \to 0$. Because $A(s\to\infty)=1$, the divergent pieces in Eqs. (B2) and (B3) can be isolated using dimensional regularization. The result is [with $a = (L/2\pi)^2 \tau_0$]

$$S_1 = \frac{1}{12}\frac{\Omega_0 T}{L^d} + \frac{1}{2}\frac{L^{2-d}}{(2\pi)^2}\left\{ I_1 - \pi^{d/2} a^{1-\epsilon/2} \times \left( \frac{2}{\epsilon} + (1-\hat{\gamma}) + O(\epsilon) \right) \right\}, \tag{B7}$$

$$S_2 = \frac{1}{2}\frac{1}{(2\pi)^2} L^{4-d}\left\{ I_2 + \pi^{d/2} a^{-\epsilon/2}\left( \frac{2}{\epsilon} - \hat{\gamma} + O(\epsilon) \right) \right\}, \tag{B8}$$

$$I_N = \int_0^\infty e^{-as}\, s^{N-1}\left\{ A(s)^4 - 1 - \left(\frac{\pi}{s}\right)^2 \right\} ds \tag{B9}$$

($\hat{\gamma} = 0.5772$, the Euler-Mascheroni constant), where the $I_N$ are finite. The two poles in $\epsilon$ can be removed by a renormalization of $\tau_0$ and $u_0$, as given in Eqs. (43) and (44).


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